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COMMENT

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Abstract. I present an operator method for the calculation of partial sums and infinite series in closed form. It consists of expressing the sum as the value at origin of a generating function which is given by a differential operator acting on a simple function. When exact expressions are not available, the method is suitable for obtaining approximate results.

Recently, Bunch *et al* [1] presented an interesting method, based on z-transforms, to calculate partial sums in closed form. They showed that

$$S_N = \sum_{n=1}^{N} x(n) = \frac{1}{2\pi \mathrm{i}} \oint \frac{x(z)(z^N - 1)}{z-1}\,\mathrm{d}z \tag{1a}$$

in which

$$x(z) = \sum_{n=0}^{\infty} x(n) z^{-n}. \tag{1b}$$

As the authors pointed out, this procedure is particularly useful when the infinite sum S_∞ diverges because one cannot treat such a case by other techniques [1]. The method of Bunch *et al* [1] relies on the assumption that the calculation of (1*b*) in closed form is simpler than the calculation of the partial sum S_N.

In what follows I present an operator method that yields an alternative expression for the partial sum S_N. The main ingredient of this method is the generating function

$$S_N(t) = \sum_{n=0}^{N} x(n) e^{nt} \tag{2}$$

which satisfies $S_N(0) = S_N$ (for the sake of simplicity I consider the sum from $n=0$ to $n = N$). The main assumption of the present method is the existence of the differential operator $x(D)$ which is a function of $D = \mathrm{d}/\mathrm{d}t$. Because e^{nt} is an eigenfunction of $x(D)$ with eigenvalue $x(n)$ one can rewrite (2) as

$$S_N(t) = x(D) \sum_{n=0}^{N} e^{nt} = x(D) \frac{e^{(N+1)t} - 1}{e^t - 1}. \tag{3}$$

In many cases the calculation of the right-hand side of the second equality in (3) is straightforward.

Before applying the operator method to illustrative examples I deem it necessary to discuss alternating sums because they contain the term $(-1)^n$ which one cannot easily write in operator form. It is convenient to factorize this term out of $x(n)$ and write

$$S_n = \sum_{n=0}^{N} (-1)^n x(n) \tag{4}$$

so that the generating function now reads

$$S_N(t) = x(D)\frac{1+(-1)^{N+1}e^{(N+1)t}}{1+e^t} \tag{5}$$

and $S_N(0) = S_N$.

In order to treat infinite sums one simply chooses $t<0$ in equations (3) or (5) and takes the limit $N \to \infty$ so that

$$S_\infty(t) = x(D)(1-e^t)^{-1} \tag{6a}$$

for the former or

$$S_\infty(t) = x(D)(1+e^t)^{-1} \tag{6b}$$

for the latter.

I illustrate the application of the expressions above by means of three examples. To begin with, I show that the operator method may enable one to obtain the function $x(z)$ introduced by Bunch *et al* [1] as follows from the fact that $S_\infty(t) = x(z)$ if $z = e^{-t}$ in equations (6). For instance, when $x(n) = n^3$ one has $x(z)$ for the example in [1]:

$$S_\infty(t) = D^3(1-e^t)^{-1} = \frac{6e^{2t}+e^t(e^t-1)^2}{(e^t-1)^4} = \frac{z(z^2+4z+1)}{(z-1)^4}. \tag{7}$$

The next example is the family of partial sums

$$S_N^{(m)} = \sum_{n=0}^{N} n^m \qquad m = 0, 1, \ldots \tag{8}$$

which according to (3) are given by

$$S_N^{(m)} = D^m \left.\frac{e^{(N+1)t}-1}{e^t-1}\right|_{t=0} \tag{9}$$

so that the calculation reduces to differentiation of a relatively simple function. Furthermore, the operator method enables one to derive a convenient recurrence relation for the partial sums $S_N^{(m)}$ because it follows from (9) that

$$\frac{e^{(N+1)t}-1}{e^t-1} = \sum_{m=0}^{\infty} S_N^{(m)} \frac{t^m}{m!}. \tag{10}$$

Therefore, on multiplying this equation by (e^t-1), expanding the expotentials on both sides of the resulting equation, collecting the coefficients of like powers of t and rearranging them conveniently one has

$$S_N^{(m)} = \frac{(N+1)^{m+1}}{m+1} - \sum_{j=2}^{m+1} \frac{m!}{j!(m+1-j)!} S_N^{(m+1-j)} \tag{11}$$

in which the sum on the right-hand side applies only if $m>0$.

Another advantage of the operator method is that it facilitates the approximate calculation of certain sums that one cannot obtain in closed form. For instance, suppose that one is interested in the behaviour of the alternating sum (4) with $x(n) = n\exp(-\theta n^2)$ for small values of θ. One can easily obtain every coefficient of the Taylor expansion about $\theta = 0$ of the partial sum from (5) with $x(D) = D^{2j+1}, j = 0, 1, \ldots$. More interesting is the infinite series which according to (6*b*) is given by

$$\sum_{n=0}^{\infty} (-1)^n n \exp(-\theta n^2) = D \exp(-\theta D^2) \left.\frac{1}{1+e^t}\right|_{t=0}. \tag{12}$$

Straightforward expansion of $\exp(-\theta D^2)$ in θ power series yields the desired approximation for small values of θ:

$$\sum_{n=0}^{\infty} (-1)^n n \exp(-\theta n^2) = -\frac{1}{4} - \frac{\theta}{8} - \frac{\theta^2}{8} - \frac{17}{96}\theta^3 - \ldots. \tag{13}$$

One cannot apply the operator method developed above to arbitrary partial sums and series and for this reason I am not presenting it as a substitute for other existing methods such as those mentioned by Bunch *et al* [1]. Rather, the operator method is complementary, having the advantage that it commonly leads to much simpler equations. Above, I discussed three examples showing that the operator method is suitable for the evaluation of sums in closed form. However, one can easily find many examples to which the operator method does not apply. One of them is $x(n) = n^p$, where p is not an integer. On shifting the subscript n by unity one is left with the operator $(1+D)^p$ which is well defined. However, one can hardly ever obtain useful expressions (even approximate ones) for the partial sum $S_N^{(p)}$. Probably other procedures do not apply to most of the cases which one cannot treat by means of the operator method.

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Reference

[1] Bunch K J, Cain W N and Grow R W 1990 *J. Phys. A: Math. Gen.* **23** L1213